

Equivalence Groups and Differential Invariants for (2+1) dimensional Nonlinear Diffusion Equation

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Abstract

(2+1) dimensional diffusion equation is considered within the framework of equivalence transformations. Generators for the group are obtained and admissible transformations between linear and nonlinear equations are examined. It is shown that transformations between linear and nonlinear equations are possible provided that the generators of independent variables depend on the dependent variable. Exact solutions for some nonlinear equations are obtained. Differential invariants related to the transformation groups are investigated and the results are compared with the direct integration method.

keywords: Lie Group Application, Equivalence Groups , Exact Solution, Non-linear Diffusion Equation, Differential Invariants

1 Introduction

Differential equations containing some arbitrary functions or parameters represent actually family of equations of the same structure. Almost all field equations of classical continuum physics possess this property related to the behaviour of different materials. In dealing with such family of differential equations, Lie symmetry analysis provides some powerful algorithmic methods for determination of invariant solutions, conserved quantities and construction of maps between differential equations of the same family that turns out to be equivalent [1–3]. To examine such problems, it is convenient considering equivalence transformation groups that preserve the structure of the family of differential equations but may change the form of the constitutive functions, parameters when appropriate transformations are available.

The first systematic treatment that the usual Lie's infinitesimal invariance approach could be employed in order to construct equivalence groups was formulated by Ovsiannikov [4]. Then several well-developed methods have been used to construct equivalence groups. The general theory of determining transformation groups and algorithms can be found in the references [4–7].

In the present text we shall examine the (2+1) dimensional diffusion equation. Nonlinear members of the family of diffusion equations have significant importance in many areas in applied sciences. A great number of members have been used to model many physical phenomena in Mathematical Physics, Mathematical Biology etc. and they have been widely studied not only by means of numerical, asymptotic analysis but also as application to Lie Group Analysis since Lie [8]. The simple nonlinear heat equation; $u_t = [A(u)u_x]_x$ was first examined by Ovsiannikov [9] within the frame-work of Lie's symmetry classification. The complete classification and form-preserving point transformations for the inhomogenous one dimensional nonlinear diffusion are obtained in [10]. The conditions for reduction the more general diffusion type equations to the one dimensional heat equation are also examined in [11]. Equivalence transformations of linear diffusion equations into nonlinear equations for some classes have been considered in Lisle's PhD thesis [12] widely. Constructing the exact analytical solutions for some specific problems related to the nonlinear diffusion equation have recently examined by [13–15]. Torrisi et al. in their paper [14] have also studied the developments of bacterial colonies as application to equivalence groups. Bruzón et al. have derived maps between nonlinear dispersive equations in their detailed work [16].

The aim of the present work is to study equivalence transformations for a general family of (2+1) dimensional diffusion equation. We investigate the structure of the transformation group generators which lead to map linear and nonlinear members.

For the convenience of the reader to follow, in section 2, we have obtained the generators of the group of equivalence transformations and determined the structure for admissible transformations. Theorems in the section state the conditions of the appropriate equivalence transformations between linear and nonlinear members. In the next section, we have considered some subgroups of the general equivalence groups and by choosing some specific forms we are able to obtain some classes of nonlinear equations which are equivalent to linear ones. We have also investigated the classes of nonlinear diffusion equations that are mapped onto the classical heat equation. Exact solutions for those nonlinear equations are also obtained. In section 4 we have also examined differential invariants for the subgroups and discussed the results with the direct integration method.

2 Equivalence Transformations

In the present paper we shall investigate the equivalence group a general family of (2+1) dimensional diffusion equation

$$u_t = f(x, y, t, u, u_x, u_y)_x + g(x, y, t, u, u_x, u_y)_y \quad (1)$$

which represents a great variety of linear and nonlinear equations. Here u is the dependent variable of the independent variables x, y and t . Here f and g are smooth nonconstant functions of their variables and subscripts denote the partial derivatives with respect to the corresponding variables.

Definition 1. *With n independent variables x_i , N dependent variables u_α and*

m smooth functions ϕ_k of independent, dependent variables and their derivatives

$$\mathcal{F}(x_i, u_{\alpha(p)}, \phi_{k(q)}(x_i, u_{\alpha(p)})) = 0$$

is called a family of differential equations. Here $i = 1, 2, \dots, n$, $\alpha = 1, 2, \dots, N$, $k = 1, 2, \dots, m$ and $u_{\alpha(p)}$ include both the tuple of dependent variables $u = (u_1, u_2, \dots, u_N)$ as well as all the derivatives of u with respect to x_i 's up to order N . By $\phi_{k(q)}$ we denote the smooth functions ϕ_k and the partial derivatives with respect to both x_i 's and $u_{\alpha(p)}$'s.

Definition 2. For a given differential equation of the family

$$\mathcal{F}(x_i, u_{\alpha(p)}, \phi_{k(q)}(x_i, u_{\alpha(p)})) = 0$$

the equivalence group \mathcal{E} is the group of smooth transformations of independent, dependent variables, their derivatives and smooth functions preserving the structure of the differential equation but transforms it into

$$\mathcal{F}(\bar{x}_i, \bar{u}_{\alpha(\bar{p})}, \bar{\phi}_{k(\bar{q})}(\bar{x}_i, \bar{u}_{\alpha(\bar{p})})) = 0.$$

More precisely, equivalence transformations associated with the (2+1) dimensional most general diffusion equation (1) transform the equation into

$$\begin{aligned} u_t - f(x, y, t, u, u_x, u_y)_x - g(x, y, t, u, u_x, u_y)_y &= 0 \longrightarrow \\ \bar{u}_{\bar{t}} - \bar{f}(\bar{x}, \bar{y}, \bar{t}, \bar{u}, \bar{u}_{\bar{x}}, \bar{u}_{\bar{y}})_{\bar{x}} - \bar{g}(\bar{x}, \bar{y}, \bar{t}, \bar{u}, \bar{u}_{\bar{x}}, \bar{u}_{\bar{y}})_{\bar{y}} &= 0. \end{aligned}$$

where $(\bar{\cdot})$ represents the transformed variables and functions.

Let $\mathcal{M} = \mathcal{N} \times \mathbb{R}$ be a (2+1) dimensional manifold with a local coordinate system $\mathbf{x} = (x_i) = (x, y, t)$ which we shall call as the space of independent variables. Consider a trivial bundle structure $(\mathcal{K}, \pi, \mathcal{M})$ with fibers are the real line \mathbb{R} . Here \mathcal{M} is the base manifold and \mathcal{K} , called the graph space is globally in form of a product manifold $\mathcal{M} \times \mathbb{R}$. We equip the four dimensional graph space \mathcal{K} with the local coordinates $(\mathbf{x}, u) = (x, y, t, u)$.

A vector field on the graph space \mathcal{K} is a section of its tangent bundle and locally in form

$$V = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \xi^3 \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u} \quad (2)$$

where ξ^i ($i = 1, 2, 3$) and η are coordinate functions on \mathcal{K} .

In order to construct the equivalence groups \mathcal{E} for the equation (1), first we extend the graph space \mathcal{K} by adding the auxiliary variables and the ones representing the functional dependencies of the smooth functions f and g

$$\tilde{\mathcal{K}} = \{x, y, t, u, f, g, u_x, u_y, u_t, f_x, f_y, f_t, f_u, g_x, g_y, g_t, g_u, f_{u_x}, f_{u_y}, g_{u_x}, g_{u_y}\}. \quad (3)$$

The prolongation vector \tilde{V} over the extended manifold covered by $\tilde{\mathcal{K}}$ can be written as

$$\begin{aligned} \tilde{V} = V &+ \mu^1 \frac{\partial}{\partial f} + \mu^2 \frac{\partial}{\partial g} + \zeta^1 \frac{\partial}{\partial u_x} + \zeta^2 \frac{\partial}{\partial u_y} + \zeta^3 \frac{\partial}{\partial u_t} + \sum_{j=1}^4 \mu_j^1 \frac{\partial}{\partial f_j} \\ &+ \sum_{j=1}^4 \mu_j^2 \frac{\partial}{\partial g_j} + \sum_{j=1}^3 \nu_j^1 \frac{\partial}{\partial f_{u_j}} + \sum_{j=1}^3 \nu_j^2 \frac{\partial}{\partial g_{u_j}} \end{aligned} \quad (4)$$

where in the last four summations $j = 1, \dots, 4$ represent x, y, t and u and all coefficient functions are smooth functions of the coordinates of the extended manifold.

Theorem 1 ([7]). *Let a vector field on an n dimensional differentiable manifold M be given by*

$$V(p) = v^i(\mathbf{x}) \frac{\partial}{\partial x^i}, \quad p = \varphi^{(-1)}(\mathbf{x}), \quad i = 1, \dots, n$$

where (U, φ) is the chart to which $p \in M$ belongs. A curve γ is an integral curve of the vector field V iff the coordinate functions $x^i(t)$ are solutions of the following system of local ordinary differential equations in \mathbb{R}^n

$$\frac{dx^i}{dt} = v^i(\mathbf{x}(t))$$

Precisely, the equivalence transformations can be determined by solving the following system of autonomous ordinary differential equations on the extended manifold (3)

$$\begin{aligned} \frac{d\bar{x}}{d\epsilon} &= \xi^1(\bar{x}, \bar{y}, \bar{t}, \bar{u}), & \frac{d\bar{y}}{d\epsilon} &= \xi^2(\bar{x}, \bar{y}, \bar{t}, \bar{u}), & \frac{d\bar{t}}{d\epsilon} &= \xi^3(\bar{x}, \bar{y}, \bar{t}, \bar{u}), \\ \frac{d\bar{u}}{d\epsilon} &= \eta(\bar{x}, \bar{y}, \bar{t}, \bar{u}), & \frac{d\bar{f}}{d\epsilon} &= \mu^1, & \frac{d\bar{g}}{d\epsilon} &= \mu^2 \end{aligned} \quad (5)$$

under the initial conditions

$$\bar{x}(0) = x, \quad \bar{y}(0) = y, \quad \bar{t}(0) = t, \quad \bar{u}(0) = u, \quad \bar{f}(0) = f, \quad \bar{g}(0) = g \quad (6)$$

where μ^1 and μ^2 depend on $(\bar{x}, \bar{y}, \bar{t}, \bar{u}, \bar{u}_x, \bar{u}_y, \bar{u}_t, \bar{f}, \bar{g})$.

Coefficients of the prolonged vector field (4), namely the infinitesimal generators for the equivalence group can be evaluated by the very well-known prolongation formula (for detailed information, theorems and detailed applications see [5, 17]). For more details we refer the reader the papers [18–20] and references therein which are concerned with equivalence transformations.

$$\begin{aligned} \zeta^1 &= D_x(\eta) - u_x D_x(\xi^1) - u_y D_x(\xi^2) - u_t D_x(\xi^3), \\ \zeta^2 &= D_y(\eta) - u_x D_y(\xi^1) - u_y D_y(\xi^2) - u_t D_y(\xi^3), \\ \zeta^3 &= D_t(\eta) - u_x D_t(\xi^1) - u_y D_t(\xi^2) - u_t D_t(\xi^3) \end{aligned} \quad (7)$$

where D_x, D_y, D_t denote the total derivatives with respect to their parameters: $D_i = \frac{\partial}{\partial x_i} + u_{x_i} \frac{\partial}{\partial u}$. And

$$\begin{aligned} \mu_j^1 &= \tilde{D}_j(\mu^1) - \sum_{i=1}^3 f_i \tilde{D}_j(\xi^i) - f_u \tilde{D}_j(\eta) - \sum_{i=1}^3 f_{u_i} \tilde{D}_j(\zeta^i), \\ \mu_j^2 &= \tilde{D}_j(\mu^2) - \sum_{i=1}^3 g_i \tilde{D}_j(\xi^i) - g_u \tilde{D}_j(\eta) - \sum_{i=1}^3 g_{u_i} \tilde{D}_j(\zeta^i), \\ \nu_j^1 &= \tilde{D}_{u_j}(\mu^1) - \sum_{i=1}^3 f_{u_i} \tilde{D}_{u_j}(\zeta^i), \\ \nu_j^2 &= \tilde{D}_{u_j}(\mu^2) - \sum_{i=1}^3 g_{u_i} \tilde{D}_{u_j}(\zeta^i) \end{aligned} \quad (8)$$

where $\tilde{D}_j = \frac{\partial}{\partial x_j}$ and $\tilde{D}_{u_j} = \frac{\partial}{\partial u_j}$.

These expressions do not impose any restriction on functional dependencies of the smooth functions f and g in the main equation (1). If some variables do not appear in the coordinate cover of the extended manifold (3), due to a particular structure of the given differential equation, that might entail some restrictions on the extended vector field components because the corresponding components must then be set to zero. Note that in equation (1) the free parameters f and g do not depend on u_t . Thus their corresponding components in (4) must vanish:

$$\mu_3^1 = \mu_3^2 = 0. \quad (9)$$

Theorem 2. *A nonlinear (2+1) dimensional diffusion equation can be mapped onto a linear equation by a point equivalence transformation, if and only if it is in the following form:*

$$\bar{x} = \phi(x, y, t, u) \quad \text{and/or} \quad \bar{y} = \psi(x, y, t, u)$$

where $\phi, \psi \in C^2$ and $\frac{\partial \phi}{\partial u} \neq 0$, $\frac{\partial \psi}{\partial u} \neq 0$.

Proof. To generate the transformations of the equivalence group for the diffusion equation given by (1), we shall apply the restrictions (9) to the given formulas (7) and (8). Then we have

$$\begin{aligned} \xi^1 &= \xi^1(x, y, t, u), \quad \xi^2 = \xi^2(x, y, t, u), \quad \xi^3 = \xi^3(t), \quad \eta = \eta(x, y, t, u), \\ \zeta^1 &= \eta_x + (\eta_u + \xi_x^1)u_x + \xi_u^1(u_x)^2 + \xi_x^2 u_y + \xi_u^2 u_x u_y, \\ \zeta^2 &= \eta_y + (\eta_u + \xi_y^1)u_y + \xi_y^1 u_x + \xi_u^1 u_x u_y + \xi_u^2 (u_y)^2, \\ \zeta^3 &= \eta_t + (\eta_u + \xi^3)u_t + \xi_t^1 u_x + \xi_u^1 u_x u_t + \xi_t^2 u_y + \xi_u^2 u_y u_t, \\ \mu^1 &= (\eta_u + \xi^3 - \xi_x^1 + \xi_u^2 u_y)f - (\xi_y^1 + \xi_u^2 u_y)g + \gamma u_y + \kappa^1, \\ \mu^2 &= (\eta_u + \xi^3 - \xi_y^2 + \xi_u^1 u_x)g - (\xi_x^2 + \xi_u^2 u_x)f - \gamma u_x + \kappa^2 \end{aligned} \quad (10)$$

and

$$\eta_u = \xi_x^1 + \xi_y^2 - \xi^3 + s(t) \quad (11)$$

where $s(t)$ is an arbitrary continuous function, γ , κ^1 and κ^2 depend on the independent and dependent variables and satisfy the following relations

$$\xi_t^1 = -\gamma_y + \kappa_u^1, \quad \xi_t^2 = \gamma_x + \kappa_u^2, \quad \eta_t = \kappa_x^1 + \kappa_y^2.$$

Equation (11) points out that η can not depend on u nonlinearly, unless ξ^1 and/or ξ^2 depend on u . Thus to have any transformation between linear and nonlinear equations ξ^1 and/or ξ^2 must involve the dependent variable u which meant to be the transformed variables \bar{x} and /or \bar{y} involve u :

$$\xi^1 = \xi^1(x, y, t, u), \quad \text{and/or} \quad \xi^2 = \xi^2(x, y, t, u), \quad \frac{\partial \xi^i}{\partial u} \neq 0, \quad i = 1, 2.$$

Thus transformed variables are obtained via the solution of the set of ordinary differential equations (5) as

$$\bar{x} = \phi(x, y, t, u) \quad \text{and/or} \quad \bar{y} = \psi(x, y, t, u).$$

□

Maps between nonlinear and linear equations might sometimes be generated by some nonlinear dependencies on only the dependent variable; like the very well-known wave equation (see [21–23]). The following theorem points out such transformations are not admissible for diffusion equation.

Theorem 3. *(2+1) dimensional diffusion equation does not admit the following type of transformation:*

$$\bar{x} = \psi_1(x, y, t), \quad \bar{y} = \psi_2(x, y, t), \quad \bar{t} = \psi_3(t), \quad \bar{u} = \psi_4(x, y, t, u)$$

where $\psi_4(x, y, t, u)$ is nonlinear in u .

Proof. One can easily see from the equation (11), unless ξ^1 or ξ^2 depend on u , η can not involve u . Thus \bar{u} can not be nonlinear in u when \bar{x} or \bar{y} does not involve u . \square

Corollary 1. *Any integrable transformation related to the generators; $\xi^1 = \xi^1(x, y, t)$, $\xi^2 = \xi^2(x, y, t)$, $\xi^3 = \xi^3(t)$, $\eta = h(x, y, t)u$ map a linear equation onto another linear equation with various different coefficient functions.*

Readers may see that equations (10) generate all set of admissible equivalence transformations, related to the general (2+1)dimensional diffusion equation (1). The equivalence transformations for some particular members of the family of (1) can be evaluated from these by integrating the system of equations (5).

3 Applications to exact solutions of nonlinear diffusion equations

Theorem 2 also refers that the admissible equivalence transformations related to the diffusion equation (1) may involve some arbitrary functions. As a consequence of this result, in addition to the maps between single linear and nonlinear equations, one can say that transformations between a single linear equation and a class of nonlinear equations can also be constructed. That is to say we can set such transformations which map a single linear equation into a particular family of nonlinear equations or vice versa.

$$f_x + g_y - u_t = 0 \longleftrightarrow \bar{f}_{k_{\bar{x}}} + \bar{g}_{k_{\bar{y}}} - \bar{u}_{k_{\bar{t}}} = 0$$

where the subscript k denotes the class of equations.

In this section we shall investigate some applications to that circumstance and study such maps by considering the transformations involving some arbitrary differentiable functions. Hereby solutions of some particular family of nonlinear equations can be obtained from an appropriate linear equation.

3.1 On the subgroup: $\xi^1 = m(u)$, $\xi^2 = h(u)$, $\xi^3 = \eta = 0$

Here we shall examine a subgroup of the admissible equivalence groups generated by the infinitesimal generators: $\xi^1 = m(u)$, $\xi^2 = h(u)$, $\xi^3 = \eta = 0$ where $m(u)$

and $h(u)$ are arbitrary differentiable continuous functions. Then the prolonged vector field (4) can be written via (10) as follows

$$\tilde{V} = m(u)\frac{\partial}{\partial x} + h(u)\frac{\partial}{\partial y} + (h'(u)f - m'(u)g)u_y\frac{\partial}{\partial f} + (m'(u)g - h'(u)f)u_x\frac{\partial}{\partial g} + \dots \quad (12)$$

By integrating the system of equations (5) under the initial conditions (6) we have the following class of transformations for the subgroup

$$\begin{aligned} \bar{x} &= x - \epsilon m(u), \quad \bar{y} = y - \epsilon h(u), \quad \bar{t} = t, \quad \bar{u} = u, \\ \bar{u}_x &= \frac{u_x}{1 - \epsilon(u_x m'(u) + u_y h'(u))}, \quad \bar{u}_y = \frac{u_y}{1 - \epsilon(u_x m'(u) + u_y h'(u))}, \\ \bar{u}_t &= \frac{u_t}{1 - \epsilon(u_x m'(u) + u_y h'(u))}, \\ \bar{f} &= \frac{(1 - \epsilon u_x m'(u))f - \epsilon u_y m'(u)g}{1 - \epsilon(u_x m'(u) + u_y h'(u))}, \quad \bar{g} = \frac{(1 - \epsilon u_y h'(u))g - \epsilon u_x h'(u)f}{1 - \epsilon(u_x m'(u) + u_y h'(u))}. \end{aligned} \quad (13)$$

Note that here ϵ is the group parameter. By substituting u_x and u_y in terms of the transformed variables, \bar{f} and \bar{g} can now be written as

$$\bar{f} = (1 + \epsilon \bar{u}_{\bar{y}} h'(\bar{u})) \tilde{f} - \epsilon \bar{u}_{\bar{y}} m'(\bar{u}) \tilde{g}, \quad \bar{g} = (1 + \epsilon \bar{u}_{\bar{x}} m'(\bar{u})) \tilde{g} - \epsilon \bar{u}_{\bar{x}} h'(\bar{u}) \tilde{f} \quad (14)$$

where $\tilde{f}(\bar{x}, \bar{y}, \bar{t}, \bar{u}, \bar{u}_{\bar{x}}, \bar{u}_{\bar{y}})$ and $\tilde{g}(\bar{x}, \bar{y}, \bar{t}, \bar{u}, \bar{u}_{\bar{x}}, \bar{u}_{\bar{y}})$ represent f and g in terms of the transformed variables.

It is clear that for every different choice of $m(u)$, $h(u)$ and linear functions f and g in the dependent variable u and its derivatives, the transformations (14) map a single linear differential equation $f_x + g_y - u_t = 0$ onto a class of nonlinear equations of the form $\bar{f}_{\bar{x}\bar{x}} + \bar{g}_{\bar{y}\bar{y}} - \bar{u}_{\bar{t}} = 0$, and a solution of the linear equation

$$\phi(x, y, t, u) = 0$$

generates solutions to the corresponding nonlinear equations as

$$\phi_k(\bar{x} + \epsilon m(\bar{u}), \bar{y} + \epsilon h(\bar{u}), \bar{t}, \bar{u}) = 0.$$

Example 1: As a particular example, we search the nonlinear diffusion equations of the class (1) that can be mapped onto the very well known heat equation. The (2+1) dimensional heat equation can be represented as a member of (1) by choosing $f = u_x$, $g = u_y$;

$$u_{xx} + u_{yy} = u_t. \quad (15)$$

The transformed form of $f = u_x$ and $g = u_y$ can be obtained from (14) as

$$\bar{f} = \frac{(1 + \epsilon h'(\bar{u}) \bar{u}_{\bar{y}}) \bar{u}_{\bar{x}} - \epsilon m'(\bar{u}) \bar{u}_{\bar{y}}^2}{1 + \epsilon h'(\bar{u}) \bar{u}_{\bar{y}} + \epsilon m'(\bar{u}) \bar{u}_{\bar{x}}}, \quad \bar{g} = \frac{(1 + \epsilon m'(\bar{u}) \bar{u}_{\bar{x}}) \bar{u}_{\bar{y}} - \epsilon h'(\bar{u}) \bar{u}_{\bar{x}}^2}{1 + \epsilon h'(\bar{u}) \bar{u}_{\bar{y}} + \epsilon m'(\bar{u}) \bar{u}_{\bar{x}}}$$

which generate the following class of nonlinear equations

$$A \bar{u}_{\bar{x}\bar{x}} + B \bar{u}_{\bar{x}\bar{y}} + C \bar{u}_{\bar{y}\bar{y}} + D = E \bar{u}_{\bar{t}} \quad (16)$$

where the coefficients are

$$\begin{aligned}
A &= 1 + 2\epsilon h'(\bar{u})\bar{u}_{\bar{y}} + \epsilon^2 (h'(\bar{u})^2 + m'(\bar{u})^2) \bar{u}_{\bar{y}}^2 \\
B &= -2\epsilon [m'(\bar{u})\bar{u}_{\bar{y}} + h'(\bar{u})\bar{u}_{\bar{x}} + \epsilon (h'(\bar{u})^2 + m'(\bar{u})^2) \bar{u}_{\bar{y}}\bar{u}_{\bar{x}}] \\
C &= 1 + 2\epsilon m'(\bar{u})\bar{u}_{\bar{x}} + \epsilon^2 (h'(\bar{u})^2 + m'(\bar{u})^2) \bar{u}_{\bar{x}}^2 \\
D &= -\epsilon (m''(\bar{u})\bar{u}_{\bar{x}} + h''(\bar{u})\bar{u}_{\bar{y}}) (\bar{u}_{\bar{x}}^2 + \bar{u}_{\bar{y}}^2) \\
E &= (1 + \epsilon(m'(\bar{u})\bar{u}_{\bar{x}} + h'(\bar{u})\bar{u}_{\bar{y}}))^2
\end{aligned} \tag{17}$$

It is obvious that via the transformations of (13), we can construct various maps between the single heat equation (15) and the nonlinear equations of the class of (16). And any solution of the heat equation (15) would generate solutions of nonlinear equations.

For the reader to see more clearly how a solution to nonlinear problems can be constructed via equivalence transformations we consider the following problem.

Example 2: A subclass of nonlinear diffusion equations (16) can be considered by taking $m(u) = u^n$, $h(u) = u^r$ in the previous example, where $n, r \in \mathbb{R}$.

$$\begin{aligned}
& \left[1 + 2\epsilon r u^{r-1} u_y + \epsilon^2 (r^2 u^{2(r-1)} + n^2 u^{2(n-1)}) u_y^2 \right] u_{xx} - 2\epsilon [n u^{n-1} u_y \\
& + r u^{r-1} u_x + \epsilon (r^2 u^{2(r-1)} + n^2 u^{2(n-1)}) u_x u_y] u_{xy} + [1 + 2\epsilon n u^{n-1} u_x \\
& + \epsilon^2 (r^2 u^{2(r-1)} + n^2 u^{2(n-1)}) u_x^2] u_{yy} - \epsilon (u_x^2 + u_y^2) (r(r-1) u^{r-2} u_y \\
& + n(n-1) u^{n-2} u_x) = (1 + \epsilon (n u^{n-1} u_x + r u^{r-1} u_y))^2.
\end{aligned} \tag{18}$$

Here $(\bar{*})$'s are omitted for the simplicity. And an implicit solution for (18) can be written as

$$u - \bar{F} \sin(\mu(x + \epsilon u^n)) \sin(\eta(y + \epsilon u^r)) e^{-(\mu^2 + \eta^2)t} = 0$$

from a solution like $u = F \sin(\mu x) \sin(\eta y) e^{-(\mu^2 + \eta^2)t}$ of the heat equation (15) by applying the equivalence transformations; $x = \bar{x} + \epsilon \bar{u}^n$, $y = \bar{y} + \epsilon \bar{u}^r$, $t = \bar{t}$, $u = \bar{u}$. Here F and \bar{F} are some constants.

Example 3: For the reader to follow the procedure better and check the calculations, let us consider another particular member of the nonlinear equation (16) by taking $m(u) = u^2$ and $h(u) = 0$, where we could simply write

$$\begin{aligned}
& (1 + \bar{u}^2 \bar{u}_{\bar{y}}^2) \bar{u}_{\bar{x}\bar{x}} - 2(1 + \bar{u} \bar{u}_{\bar{x}}) \bar{u} \bar{u}_{\bar{y}} \bar{u}_{\bar{x}\bar{y}} + (1 + \bar{u} \bar{u}_{\bar{x}})^2 \bar{u}_{\bar{y}\bar{y}} - (\bar{u}_{\bar{y}}^2 + \bar{u}_{\bar{x}}^2) \bar{u}_{\bar{x}} \\
& = (1 + \bar{u} \bar{u}_{\bar{x}})^2 \bar{u}_{\bar{t}}.
\end{aligned} \tag{19}$$

The reader can easily see, a much more easier function $u = x + y^2 + 2t$ which satisfies the linear heat equation would generate

$$\bar{u} = 1 \mp \sqrt{1 - 2(\bar{x} + \bar{y}^2 + 2\bar{t})}$$

as a solution to the nonlinear equation (19) under the transformations $x = \bar{x} + \frac{1}{2} \bar{u}^2$, $y = \bar{y}$, $t = \bar{t}$, $u = \bar{u}$.

Example 4: As another example by choosing $m(u) = \sin u$, $h(u) = \cos u$ the

nonlinear equation would become

$$\begin{aligned}
& (1 + \bar{u}_{\bar{y}}^2 - 2\bar{u}_{\bar{y}} \sin \bar{u})\bar{u}_{\bar{x}\bar{x}} + (1 + \bar{u}_{\bar{x}}^2 + 2\bar{u}_{\bar{x}} \cos \bar{u})\bar{u}_{\bar{y}\bar{y}} \\
& - 2(\bar{u}_{\bar{y}} \cos \bar{u} - \bar{u}_{\bar{x}} \sin \bar{u} + \bar{u}_{\bar{x}}\bar{u}_{\bar{y}})\bar{u}_{\bar{x}\bar{y}} + (\bar{u}_{\bar{y}} \cos \bar{u} + \bar{u}_{\bar{x}} \sin \bar{u})(\bar{u}_{\bar{y}}^2 + \bar{u}_{\bar{x}}^2) \\
& - (1 - \bar{u}_{\bar{y}} \sin \bar{u} + \bar{u}_{\bar{x}} \cos \bar{u})^2 \bar{u}_{\bar{t}} = 0.
\end{aligned} \tag{20}$$

can be transformed into the heat equation (15) by the equivalence transformations: $\bar{x} = x - \sin u$, $\bar{y} = y - \cos u$, $\bar{u} = u$. Any solution of the linear heat equation would generate a solution of the equation.

3.2 On the subgroup: $\xi^1 = m(u, y)$, $\xi^2 = \xi^3 = \eta = 0$

Similarly to the previous subsection, transformations related to a subgroup generated by the infinitesimal generators $\xi^1 = m(u, y)$, $\xi^2 = \xi^3 = \eta = 0$ are considered. The corresponding equivalence transformations of the subgroup are explicitly determined as:

$$\begin{aligned}
\bar{x} &= x - \epsilon m(u, y), \quad \bar{y} = y, \quad \bar{t} = t, \quad \bar{u} = u, \\
\bar{u}_x &= \frac{u_x}{1 - \epsilon m_u(u, y)u_x}, \quad \bar{u}_y = \frac{u_y + \epsilon m_y(u, y)u_x}{1 - \epsilon m_u(u, y)u_x}, \quad \bar{u}_t = \frac{u_t}{1 - \epsilon m_u(u, y)u_x}, \\
\bar{f} &= f - \frac{\epsilon(m_y(u, y) + m_u(u, y)u_y)g}{1 - \epsilon m_u(u, y)u_x}, \quad \bar{g} = \frac{g}{1 - \epsilon m_u(u, y)u_x}
\end{aligned}$$

where subscripts indicate partial differentiation with respect to the corresponding variable.

Example 5: For the simplicity, let us again seek the nonlinear diffusion equations that can be mapped onto the linear heat equation (15). For $f = u_x$ and $g = u_y$, \bar{f} and \bar{g} turn to be:

$$\bar{f} = \frac{\bar{u}_{\bar{x}} - \epsilon(m_{\bar{y}} + m_{\bar{u}}\bar{u}_{\bar{y}})(\bar{u}_{\bar{y}} - \epsilon m_{\bar{y}}\bar{u}_{\bar{x}})}{1 + \epsilon m_{\bar{u}}\bar{u}_{\bar{x}}}, \quad \bar{g} = \bar{u}_{\bar{y}} - \epsilon m_{\bar{y}}\bar{u}_{\bar{x}}$$

where $m = m(\bar{u}, \bar{y})$. Thus the following class are the class of nonlinear equations that mapped onto the heat equation:

$$A \bar{u}_{\bar{x}\bar{x}} + B \bar{u}_{\bar{x}\bar{y}} + C \bar{u}_{\bar{y}\bar{y}} + D = E \bar{u}_{\bar{t}} \longrightarrow u_{xx} + u_{yy} = u_t$$

where the coefficients are

$$\begin{aligned}
A &= 1 + \epsilon^2 m_{\bar{y}}^2 + 3\epsilon^2 m_{\bar{u}} m_{\bar{y}} \bar{u}_{\bar{y}} + 2\epsilon^2 m_{\bar{u}}^2 \bar{u}_{\bar{y}}^2 \\
B &= \epsilon [\epsilon^2 m_{\bar{u}}^2 \bar{u}_{\bar{x}}^2 (m_{\bar{u}} \bar{u}_{\bar{y}} + m_{\bar{y}}) - 2\epsilon m_{\bar{u}}^2 \bar{u}_{\bar{x}} \bar{u}_{\bar{y}} - \epsilon m_{\bar{u}} m_{\bar{y}} \bar{u}_{\bar{x}} - 3m_{\bar{u}} \bar{u}_{\bar{y}} - 2m_{\bar{y}}] \\
C &= 1 + \epsilon m_{\bar{u}} \bar{u}_{\bar{x}} - \epsilon^2 m_{\bar{u}}^2 \bar{u}_{\bar{x}}^2 - \epsilon^3 m_{\bar{u}}^3 \bar{u}_{\bar{x}}^3 \\
D &= \epsilon [2\epsilon^2 m_{\bar{u}} m_{\bar{y}} m_{\bar{u}\bar{y}} - \epsilon^2 m_{\bar{u}}^2 m_{\bar{y}\bar{y}} - m_{\bar{u}\bar{u}} - \epsilon^2 m_{\bar{y}}^2 m_{\bar{u}\bar{u}}] \bar{u}_{\bar{x}}^3 + \epsilon^2 [-2m_{\bar{y}\bar{y}} m_{\bar{u}} \\
& + 2m_{\bar{u}\bar{y}} m_{\bar{y}} - 3m_{\bar{u}\bar{y}} m_{\bar{u}} \bar{u}_{\bar{y}} + 3m_{\bar{u}\bar{u}} m_{\bar{y}} \bar{u}_{\bar{y}}] \bar{u}_{\bar{x}}^2 - 2\epsilon m_{\bar{u}\bar{u}} \bar{u}_{\bar{x}} \bar{u}_{\bar{y}}^2 \\
& - 3\epsilon m_{\bar{u}\bar{y}} \bar{u}_{\bar{x}} \bar{u}_{\bar{y}} - \epsilon m_{\bar{y}\bar{y}} \bar{u}_{\bar{x}} \\
E &= (1 + \epsilon m_{\bar{u}} \bar{u}_{\bar{x}})^2
\end{aligned} \tag{21}$$

Any solution $\phi(x, y, t, u) = 0$ of the heat equation (15) generates hereby the solution $\phi(\bar{x} + \epsilon m(\bar{u}, \bar{y}), \bar{y}, \bar{t}, \bar{u}) = 0$ for the nonlinear equation (16) with the coefficients (21).

3.3 On the subgroup: $\xi^2 = m(u, x)$, $\xi^1 = \xi^3 = \eta = 0$

Completely similar to the previous subsections, the infinitesimal generators on the vector field of (4) yield the transformations

$$\begin{aligned}\bar{x} &= x, & \bar{y} &= y - \epsilon m(u, x), & \bar{t} &= t, & \bar{u} &= u, \\ \bar{u}_x &= \frac{u_x + \epsilon m_x(u, x)u_y}{1 - \epsilon m_u(u, x)u_y}, & \bar{u}_y &= \frac{u_y}{1 - \epsilon m_u(u, x)u_y}, & \bar{u}_t &= \frac{u_t}{1 - \epsilon m_u(u, x)u_y}, \\ \bar{f} &= \frac{f}{1 - \epsilon m_u(u, x)u_y}, & \bar{g} &= g - \frac{\epsilon(m_x(u, x) + \epsilon m_u(u, x)u_x)g}{1 - \epsilon m_u(u, x)u_y}.\end{aligned}\quad (22)$$

As the procedure is already given in detail in the previous subsections with many examples, here we will only consider the following example which is a little different than the previous ones.

Example 6: Even though the first order PDE,

$$u_x + u_y - u_t = 0$$

is not a diffusion equation, we may still examine it under the equivalence transformations generated for the diffusion equation (1) by taking $f = g = u$. The following class of nonlinear equations are mapped onto the given constant coefficient PDE under the transformation group (22):

$$\bar{u}_{\bar{x}} + (1 - \epsilon m_{\bar{x}}(\bar{u}, \bar{x}))\bar{u}_{\bar{y}} - \bar{u}_{\bar{t}} = 0. \quad (23)$$

And a solution to the equation (23) can be written as

$$\bar{u} - \psi(\bar{t} + \bar{x}, \bar{y} + \epsilon m(\bar{u}, \bar{x}) - \bar{x}) = 0$$

from the general solution of the linear equation $u = \psi(t + x, y - x)$. Here we should warn the reader that even though the solution of the linear equation is its general solution, the transformed solution is not the general solution to the nonlinear equation.

3.4 On the subgroup: $\xi^1 = \xi^2 = \xi^3 = 0$, $\eta = m(x, y, t)$

As the last case, here we shall consider the equivalence transformations of linear or nonlinear equations which do not map between each other, but map one into another with different coefficient functions. We can construct many of such transformations, but as an example here will investigate one subgroup by taking the infinitesimal generators drive the following equivalence transformations

$$\begin{aligned}\bar{x} &= x, & \bar{y} &= y, & \bar{t} &= t, & \bar{u} &= u + \epsilon m(x, y, t), \\ \bar{u}_x &= u_x + \epsilon m_x(x, y, t), & \bar{u}_y &= u_y + \epsilon m_y(x, y, t), & \bar{u}_t &= u_t + \epsilon m_t(x, y, t), \\ \bar{f} &= f + \epsilon \int m_t(x, y, t) dx, & \bar{g} &= g.\end{aligned}$$

Under such transformations $f_x + g_y - u_t = 0$ is mapped onto

$$\left[f + \epsilon \int m_t(x, y, t) dx \right]_x + g_y - u_t = \epsilon m_t(x, y, t).$$

One can say by considering such transformations we may address the non-homogeneous equations to homogeneous equations.

It is obviously clear that one investigate more subgroups of the general equivalence groups. And for each subgroup, various different maps between linear and nonlinear diffusion equations can be examined. But we hope the subgroups and examples examined here will suffice to give some idea to the general structure on the maps between nonlinear diffusion equations and linear equations.

4 Differential Invariants

Differential Invariants of Lie groups of continuous transformations play important role in mathematical modelling, differential geometry and nonlinear field equations. In recent years, differential invariants admitting equivalence transformations have been mostly studied for constructing maps between linear and nonlinear differential equations. The reader may look at [24] and [21] for the application of differential invariants to the linearization problem for nonlinear wave equation. The linearization problem via differential invariants for one dimensional diffusion equation was investigated in [25–28]. Recently the problem is studied for third order evolution equation by Tsaousi et al. in [29]. Like the equivalence transformation for the general class of diffusion equation (1) has not been studied yet its differential invariants have not been considered in any research as well.

One can understand that determining the differential invariants for the complete group of equivalence transformations for the equation (1) is almost impossible. Thus we will here investigate the differential invariants admitting the special subgroups of equivalence transformations represented in section 3.1 and discuss about the results.

4.1 Differential Invariants for the Subgroup: $\xi^1 = m(u), \xi^2 = h(u)$

The subgroup which are examined in Section 3.1 for the class of (2+1) dimensional diffusion equation (1) is infinite dimensional and spanned by the vector fields

$$\begin{aligned} V_m &= m(u) \frac{\partial}{\partial x} + m'(u) \left[u_x^2 \frac{\partial}{\partial u_x} + u_x u_y \frac{\partial}{\partial u_y} + u_x u_t \frac{\partial}{\partial u_t} - u_y g \frac{\partial}{\partial f} + u_x g \frac{\partial}{\partial g} \right], \\ V_h &= h(u) \frac{\partial}{\partial y} + h'(u) \left[u_x u_y \frac{\partial}{\partial u_x} + u_y^2 \frac{\partial}{\partial u_y} + u_y u_t \frac{\partial}{\partial u_t} + u_y f \frac{\partial}{\partial f} - u_x f \frac{\partial}{\partial g} \right] \end{aligned} \quad (24)$$

where

$$[V_m, V_h] = V_m(V_h) - V_h(V_m) = 0.$$

Definition 3. A function

$$J = J(x, y, t, u, u_x, u_y, u_t, f, g)$$

is called the invariant of order zero of the (2+1) dimensional diffusion equation (1), if it is invariant under the equivalence groups V_m and V_h given by (24).

Invariance conditions $V_m(J) = 0$ and $V_h(J) = 0$ yield the invariant function of order zero to be

$$J = J(t, u, \xi_1, \xi_2, \xi_3)$$

where

$$\xi_1 = \frac{u_y}{u_x}, \quad \xi_2 = \frac{u_t}{u_x}, \quad \xi_3 = f \frac{1}{\xi_1} + g. \quad (25)$$

One can easily see that these differential invariants are consistent with the results (13) obtained by the direct integration method

$$\frac{u_y}{u_x} = \frac{\bar{u}_{\bar{y}}}{\bar{u}_{\bar{x}}}, \quad \frac{u_t}{u_x} = \frac{\bar{u}_{\bar{t}}}{\bar{u}_{\bar{x}}}, \quad f \frac{u_x}{u_y} + g = \bar{f} \frac{\bar{u}_x}{\bar{u}_y} + \bar{g}.$$

Definition 4. A function

$$J(x, y, t, u, u_x, u_y, u_t, f, g, f_x, f_y, f_t, f_u, f_{u_x}, f_{u_y}, g_x, g_y, g_t, g_u, g_{u_x}, g_{u_y})$$

is called the invariant of order one for the diffusion equation (1), if it is invariant under the equivalence groups related to the prolonged vector fields \tilde{V}_m and \tilde{V}_h .

Computation of the differential invariants of order one for this subgroup too complicated as they involve two free functions. For the simplicity, here we will examine the procedure finding the first order invariants by considering the subgroup in which $h(u) = 0$.

In addition to V_m given by (24), substituting the additional components of the vector field (4) which determined by (8) yields the prolongation vector \tilde{V}_m to be

$$\begin{aligned} \tilde{V}_m = & - [m'(u)(f_x + u_y g_u) + m''(u)(u_x^2 f_{u_x} + u_x u_y f_{u_y})] \frac{\partial}{\partial f_u} \\ & - [m'(u)(g_x - u_x g_u) + m''(u)(u_x^2 g_{u_x} + u_x u_y g_{u_y} - u_x g)] \frac{\partial}{\partial g_u} \\ & - m'(u) u_y g_x \frac{\partial}{\partial f_x} - m'(u) u_y g_y \frac{\partial}{\partial f_y} - m'(u) u_y g_t \frac{\partial}{\partial f_t} + m'(u) u_x g_x \frac{\partial}{\partial g_x} \\ & + m'(u) u_x g_y \frac{\partial}{\partial g_y} + m'(u) u_x g_t \frac{\partial}{\partial g_t} - m'(u)(2u_x f_{u_x} + u_y f_{u_y} + u_y g_{u_x}) \frac{\partial}{\partial f_{u_x}} \\ & - m'(u)(u_x f_{u_y} + u_y g_{u_y} + g) \frac{\partial}{\partial f_{u_y}} - m'(u)(u_x g_{u_x} + u_y g_{u_y} - g) \frac{\partial}{\partial g_{u_x}}. \end{aligned} \quad (26)$$

Since $m(u)$ is an arbitrary function, we apply the invariant test

$$V_m(J) = 0, \quad \tilde{V}_m(J) = 0$$

and obtain the invariant function of order one as a function depending on 15 invariants

$$J(y, t, u, \xi_i), \quad i = 1, 2, \dots, 13$$

where

$$\begin{aligned}
\xi_1 &= \frac{u_y}{u_x}, & \xi_2 &= \frac{u_t}{u_x}, & \xi_3 &= \frac{g}{u_x}, & \xi_4 &= f + \xi_1 g, \\
\xi_5 &= g_x + \frac{1}{\xi_1} f_x, & \xi_6 &= \frac{g_y}{g_x} \frac{1}{\xi_1}, & \xi_7 &= f_y - \frac{g_y}{g_x} f_x, \\
\xi_8 &= \frac{g_t}{g_x} \frac{1}{\xi_1}, & \xi_9 &= f_t - \frac{g_t}{g_x} f_x, & \xi_{10} &= g_{u_y}, \\
\xi_{11} &= -(\xi_1 \xi_{10} + \xi_3) f_x + f_{u_y} g_x \xi_1, & \xi_{12} &= (\xi_3 - \xi_1 g_{u_y}) f_x + \xi_1 g_x g_{u_x}, \\
\xi_{13} &= [(g_{u_y} f_x - g_{u_x} g_x) f_x + (f_{u_x} g_x - f_{u_y} f_x) g_x] \xi_1^2.
\end{aligned} \tag{27}$$

A symbolic software is used to compute the differential invariants. Differential invariants related to the prolonged vector field \tilde{V}_h can also be obtained but because the procedure is the same we neglect that part.

5 Conclusion and Remarks

In this work we considered the equivalence transformations for a general (2+1) dimensional diffusion equation with no restriction on the functional dependencies of free functions. The goal was to construct the most general infinitesimal generators for the transformations associated equivalence group and investigate the structure of admissible transformations between linear and nonlinear equations. We showed that such transformations were only possible when the transformed independent variables involve the dependent variable. Similar analysis can either be applied to some smaller classes directly or the results obtained here can be applied by appropriate restrictions.

Second, we considered some subgroups and chose some particular transformations to generate maps between linear and nonlinear equations and we determined the class of nonlinear diffusion equations that can be mapped onto the linear heat equation. We have not interested in algebraic structure and the classification problem. Classification for some members of the diffusion equation can be a subject of another study.

Third, in the last section we investigated the differential invariants of order zero and of order one for a subgroup which was examined in the previous section. And were able to show that the zeroth order differential invariants were compatible with the results we obtained by direct integration method. The determination of the differential invariants can easily be extended by taking some other particular subgroups by running the similar calculations.

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